Finding the Maximum Duration of Constant Positive Consumption in a Closed Economy Dependent on a Non-renewable Resource

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The Problem

The output of a closed economy consists of a quantity of a single good, any part of which is immediately either consumed or added to the stock of produced capital. Once added to the stock of produced capital it cannot subsequently be consumed. The production function is:

$$Y(t) = A(K(t))^{\alpha}(R(t))^{\beta} \qquad (A, \alpha, \beta > 0;\ \alpha + \beta < 1)$$

where:

$Y(t) =$ output at time t;

$K(t) =$ stock of produced capital at time t;

$R(t) =$ rate of use of a non-renewable resource at time t;

A, α, β are fixed parameters, the value of A reflecting the technology and the labour input, both of which are assumed constant.

The stock of produced capital is subject to depreciation at a rate $\delta K(t)$ $(0 < \delta < 1)$.

Given positive initial stocks K_0 of produced capital and S_0 of the resource, what is the maximum time over which consumption can be maintained at a constant given rate C, and what time path of R is needed to achieve that maximum time?

Formulation as an Optimal Control Problem

Let T be the duration of consumption at rate C, We have to maximise T but, to fit the standard format of an optimal control problem it is convenient to write this as an integral:

$$\max \int_o^T 1\, dt$$

There are two state variables, the stock of produced capital K and the stock of the resource S (I will sometimes omit the time variable where it is clear from the context). The control variable is R. The equations of motion are:

$$\dot{K} = AK^{\alpha}R^{\beta} - C - \delta K$$

$$\dot{S} = -R$$

There is also an inequality constraint: output must be sufficient to provide for consumption.

$$C - AK^\alpha R^\beta \leq 0$$

While K and R must be positive, separate constraints are not needed since it can be shown that they are implicit in the other conditions. A further constraint is that S must be non-negative, but since S must gradually decrease from its initial value, the constraint will only become binding at $t = T$ and can be handled by setting $S(T) = 0$ as a terminal condition, so that the full set of conditions for a state-space constraint are not needed. Terminal capital, $K(T)$, is free.

The given initial conditions are $K = K_0$ and $S = S_0$.

Necessary Conditions for a Solution

Introducing the co-state variables λ for K and μ for S, the Hamiltonian is:

$$H = 1 + \lambda(AK^\alpha R^\beta - C - \delta K) + \mu(-R)$$

Given the inequality constraint, and since a constraint qualification (Chiang p 278) is satisfied by the fact that (with $0 < \beta < 1$) $AK^\alpha R^\beta$ is convex in R and therefore $C - AK^\alpha R^\beta$ is concave in R, we need to expand this to a Lagrangian including an additional co-state variable ν:

$$L = 1 + \lambda(AK^\alpha R^\beta - C - \delta K) + \mu(-R) + \nu(AK^\alpha R^\beta - C)$$

The necessary conditions are:

$$\frac{\partial L}{\partial R} = (\lambda + \nu)AK^\alpha \beta R^{\beta-1} - \mu = 0 \qquad (1)$$

$$\dot{\lambda} = -\frac{\partial L}{\partial K} = -(\lambda + \nu)A\alpha K^{\alpha-1}R^\beta + \lambda\delta \qquad (2)$$

$$\dot{\mu} = \frac{\partial L}{\partial S} = 0 \qquad (3)$$

$$\dot{K} = \frac{\partial L}{\partial \lambda} = AK^\alpha R^\beta - C - \delta K \qquad (4)$$

$$\dot{S} = \frac{\partial L}{\partial \mu} = -R \qquad (5)$$

and the complementary slackness conditions (Chiang p 279) arising from the inequality constraint:

$$\nu \geq 0 \qquad (6) \qquad\qquad \nu(AK^\alpha R^\beta - C) = 0 \qquad (7)$$

The terminal conditions are:

$$\lambda(T) = 0 \qquad (8) \qquad\qquad S(T) = 0 \qquad (9)$$

Inferences from the Necessary Conditions: Case $\nu > 0$

From (7) we can infer:

$$AK^{\alpha}R^{\beta} - C = 0 \qquad (10)$$

Given (10), the equation of motion (4) reduces to:

$$\dot{K} = -\delta K \qquad (11)$$

This is readily solved (eg via the standard substitution $K = e^{mt}$), and writing n for the time at the start of an interval over which $\nu > 0$ the time path of K over that interval is:

$$K = K(n)e^{-\delta(t-n)} \qquad (12)$$

Rearranging (10) and using (12) to substitute for K we can also obtain the time path of R over the same interval:

$$R = C^{1/\beta}A^{-1/\beta}K^{-\alpha/\beta} = C^{1/\beta}A^{-1/\beta}(K(n))^{-\alpha/\beta}e^{\alpha\delta(t-n)/\beta} \qquad (13)$$

Suppose now that the interval over which $\nu = 0$ continues until $t = T$, that is, until the stock of resource at time n, $S(n)$, has been exhausted. To find the length of the interval $[n, T]$, we set that stock equal to the integral of R over that interval:

$$S(n) = \int_n^T R\,dt = \int_n^T C^{1/\beta}A^{-1/\beta}(K(n))^{-\alpha/\beta}e^{\alpha\delta(t-n)/\beta}\,dt \qquad (14)$$

Evaluating the integral after taking the constant terms outside:

$$S(n) = C^{1/\beta}A^{-1/\beta}(K(n))^{-\alpha/\beta}\int_n^T e^{\alpha\delta(t-n)/\beta}\,dt \qquad (15)$$

$$S(n) = C^{1/\beta}A^{-1/\beta}(K(n))^{-\alpha/\beta}\left[\frac{\beta}{\alpha\delta}e^{\alpha\delta(t-n)/\beta}\right]_n^T \qquad (16)$$

$$S(n) = \frac{\beta C^{1/\beta}A^{-1/\beta}(K(n))^{-\alpha/\beta}}{\alpha\delta}(e^{\alpha\delta(T-n)/\beta} - 1) \qquad (17)$$

$$\frac{\alpha\delta S(n)A^{1/\beta}(K(n))^{\alpha/\beta}}{\beta C^{1/\beta}} + 1 = e^{\alpha\delta(T-n)/\beta} \qquad (18)$$

Hence the length of time over which the stock of resource at time n will be exhausted is given by:

$$T - n = \frac{\beta}{\alpha\delta}\ln\Big(\frac{\alpha\delta S(n)A^{1/\beta}(K(n))^{\alpha/\beta}}{\beta C^{1/\beta}} + 1\Big) \qquad (19)$$

or equivalently:

$$T - n = \frac{\beta}{\alpha\delta}\ln\Big(\frac{\alpha\delta S(n)A^{1/\beta}(K(n))^{\alpha/\beta} + \beta C^{1/\beta}}{\beta C^{1/\beta}}\Big) \qquad (20)$$

Note that (14) to (20) all depend on the supposition that $\nu = 0$ continues until $t = T$.

A Lower Bound on the Maximum Duration

Although there is no reason to expect that the equations of motion (12) and (13) will in all circumstances maximise the duration of consumption at the rate C, they do enable us, since the option of following those equations is always available, to formulate a *lower bound* on that maximum duration, in other words, a minimum time over which consumption at rate C can be maintained. Putting $n = 0$ in (20) and (because the equations do not imply optimality when $\nu = 0$) replacing $=$ by $\geq$, we have:

Proposition 1a *A lower bound on the maximum duration T is given by:*

$$T \geq \frac{\beta}{\alpha\delta} \ln \left(\frac{\alpha\delta S_0 A^{1/\beta} K_0^{\alpha/\beta} + \beta C^{1/\beta}}{\beta C^{1/\beta}} \right)$$

Further Inferences using Economic Reasoning

There seems no obvious way, using just the necessary conditions above, of determining the range of times over which $\nu > 0$ or, in particular, of justifying the above supposition that, once that condition holds, it continues to hold until the resource has been exhausted. But further progress can be made using economic reasoning.

A marginal unit of resource at a time n when $\nu > 0$ could be applied in either of two ways. It could be added to the stock of resource $S(n)$, leaving more resource available for subsequent use. Alternatively, it could be added to the resource input at time n, $R(n)$, with the consequence that the rate of output at that time would slightly exceed C and the resulting capital stock would be marginally more than it would have been if output had equalled C. Either would extend the interval $[n, T]$, but not normally by the same amount. We will find and compare the marginal products of these two uses, where the "product" is the length of that interval.

If the marginal unit of the resource is used to add to $S(n)$, differentiating (20) with respect to S, the marginal product is:

$$\frac{d(T-n)}{d(S(n))} = \frac{\beta}{\alpha\delta} \Big(\frac{\beta C^{1/\beta}}{\alpha\delta S(n) A^{1/\beta} (K(n))^{\alpha/\beta} + \beta C^{1/\beta}} \Big) \alpha\delta A^{1/\beta} (K(n))^{\alpha/\beta} \qquad (21)$$

which simplifies to:

$$\frac{d(T-n)}{dS(n)} = \frac{\beta^2 C^{1/\beta} A^{1/\beta} (K(n))^{\alpha/\beta}}{\alpha\delta S(n) A^{1/\beta} (K(n))^{\alpha/\beta} + \beta C^{1/\beta}} \qquad (22)$$

To find the marginal product of adding to use of the resource, we must first obtain the effect on $K(n)$. From the production function, the marginal product (in terms of output) of the resource is in general:

$$\frac{dY}{dR} = AK^{\alpha} \beta R^{\beta-1} \qquad (23)$$

At time n, using (10), we have:

$$R(n) = C^{1/\beta} A^{-1/\beta} (K(n))^{-\alpha/\beta} \qquad (24)$$

Substituting (24) into (23) and noting (from the equation of motion of K) that the marginal effect on output will affect produced capital on a 1:1 basis:

$$\frac{dK(n)}{dR} = A(K(n))^{\alpha} \beta C^{(\beta-1)/\beta} A^{(1-\beta)/\beta} (K(n))^{\alpha(1-\beta)/\beta} = A^{1/\beta} \beta C^{(\beta-1)/\beta} K(n))^{\alpha/\beta} \qquad (25)$$

The required marginal product, differentiating (20) with respect to K and using (25), is then:

$$\frac{d(T-n)}{dR} = \frac{d(T-n)}{dK(n)}\frac{dK(n)}{dR} =$$

$$= \frac{\beta}{\alpha\delta}\Big(\frac{\beta C^{1/\beta}}{\alpha\delta S(n)A^{1/\beta}(K(n))^{\alpha/\beta}+\beta C^{1/\beta}}\Big)\alpha\delta S(n)A^{1/\beta}(\alpha/\beta))K(n))^{(\alpha-\beta)/\beta}A^{1/\beta}\beta C^{(\beta-1)/\beta}K(n))^{\alpha/\beta} \qquad (26)$$

$$\frac{d(T-n)}{dR} = \frac{\alpha\beta^2 CS(n)A^{2/\beta}(K(n))^{(2\alpha-\beta)/\beta}}{\alpha\delta S(n)A^{1/\beta}(K(n))^{\alpha/\beta}+\beta C^{1/\beta}} \qquad (27)$$

Comparing the marginal products (22) and (27) and focusing on their numerators since their denominators are identical, their ratio simplifies to:

$$\frac{d(T-n)/dS(n)}{d(T-n)/dR} = \frac{C^{(1-\beta)/\beta}}{\alpha S(n)A^{1/\beta}(K(n))^{(\alpha-\beta)/\beta}} \qquad (28)$$

We can infer that the marginal product of an addition to the stock of resource is no less than the marginal product of an addition to the resource input when, and only when the right hand side of (28) is greater than or equal to 1, or equivalently:

$$S(n) \leq \frac{C^{(1-\beta)/\beta}}{\alpha A^{1/\beta}(K(n))^{(\alpha-\beta)/\beta}} \qquad (29)$$

Whenever this condition holds, it is better (in terms of the objective of maximising T) to use only the minimum quantity of resource required for output to equal C, rather than using some resource to invest in produced capital. Moreover, if this condition holds at some time,it must hold at any later time since $S(n)$ always decreases, while with output equalling C, $K(n)$ decreases at the rate of depreciation and therefore the right hand side of (29) increases.

Writing n^* for the earliest time at which condition (29) holds, we can see that:

$$S(n^*) = \frac{C^{(1-\beta)/\beta}}{\alpha A^{1/\beta}(K(n^*))^{(\alpha-\beta)/\beta}} \qquad (30)$$

Thus n^* is the time at which the system shifts from $\nu = 0$ to $\nu > 0$, and having done so it never reverts to $\nu = 0$.

It will be useful below to consider the position just before n^*, at time $n^* - \epsilon$ where ϵ is an infinitesimal. During the interval [n$^* - \epsilon, n^*$], the changes in the state variables K and S must be infinitesimal. Hence the equations (21) to (28), on substituting $n^* - \epsilon$ for n, must be at worst inequalities in which the difference is no more than an infinitesimal quantity. Of particular interest is the effect on (24), implying that $R(n^* - \epsilon)$ differs by no more than an infinitesimal quantity from $R(n^*)$ and therefore $Y(n^* - \epsilon)$ differs by no more than an infinitesimal quantity from $Y(n^*) = C$. For optimality therefore we require:

$$\lim_{\epsilon\to 0} Y(n^* - \epsilon) = C \qquad (31)$$

More on the Lower Bound

From the above reasoning we can now strengthen Proposition 1 to the following:

Proposition 1b *A lower bound on the maximum duration T is given by:*

$$T \geq \frac{\beta}{\alpha\delta} \ln \left(\frac{\alpha\delta S_0 A^{1/\beta} K_0^{\alpha/\beta} + \beta C^{1/\beta}}{\beta C^{1/\beta}} \right)$$

with strict equality if:

$$S_0 \leq \frac{C^{(1-\beta)/\beta}}{\alpha A^{1/\beta} K_0^{(\alpha-\beta)/\beta}}$$

Inferences from the Necessary Conditions: Case $\nu = 0$

Differentiating (1) with respect to time and using (3) to eliminate $\dot{\mu}$:

$$\dot{\lambda} A K^{\alpha} \beta R^{\beta-1} + \lambda A K^{\alpha-1} \dot{K} \beta R^{\beta-1} + \lambda A K^{\alpha} \beta(\beta - 1) R^{\beta-2} \dot{R} = 0 \qquad (32)$$

Using (2) to substitute for $\dot{\lambda}$:

$$\lambda(-A\alpha K^{\alpha-1} R^{\beta} + \delta) A K^{\alpha} \beta R^{\beta-1} + \lambda A \alpha K^{\alpha-1} \dot{K} \beta R^{\beta-1} + \lambda A K^{\alpha} \beta(\beta - 1) R^{\beta-2} \dot{R} = 0 \qquad (33)$$

Dividing by $\lambda A K^{\alpha} \beta R^{\beta-1}$:

$$-A\alpha K^{\alpha-1} R^{\beta} + \delta + \alpha K^{-1} \dot{K} + (\beta - 1) R^{-1} \dot{R} = 0 \qquad (34)$$

Multiplying (4) by αK^{-1} and rearranging:

$$-A\alpha K^{\alpha-1} R^{\beta} + \alpha K^{-1} \dot{K} = -\alpha K^{-1} C - \alpha\delta \qquad (35)$$

Substituting from (35) into (34) and then rearranging:

$$\text{-}\alpha K^{-1} C - \alpha\delta + \delta + (\beta - 1) R^{-1} \dot{R} = 0 \qquad (36)$$

$$\dot{R} = \frac{R}{1-\beta} \left(\delta(1 - \alpha) - \frac{\alpha C}{K} \right) \qquad (37)$$

From (4) we have:

$$\dot{K} = A K^{\alpha} R^{\beta} - C - \delta K \qquad (38)$$

Considering (37) and (38) together, we have simultaneous differential equations which, in conjunction with the initial conditions, should determine the optimal paths of K and R while $\nu = 0$. However the equations appear intractable: there is no obvious way to an analytic solution.

Finding Approximate Solutions For Particular Values of the Parameters

A first step towards a method for finding an approximate solution is to write down difference equations corresponding to (37) and (38), There is more than one way in which this can be done: our approach has K_{t+1} entirely dependent on variables in period t, but R_{t+1} dependent on K_{t+1} as well as R_t.

$$R_{t+1} = R_t + \frac{R_t}{1-\beta} \left(\delta(1 - \alpha) - \frac{\alpha C}{K_{t+1}} \right) \qquad (39)$$

$$K_{t+1} = K_t + A K_t^{\alpha} R_t^{\beta} - C - \delta K_t \qquad (40)$$

Corresponding to (5) we can add a difference equation for the stock of resource S:

$$S_{t+1} = S_t - R_t \qquad (41)$$

Note that we are now working in discrete time and will take the first period of production to be period 1.

Given the values of α, β, δ, A and C, the initial state values K_1 and S_1, and a *trial* initial value R_1 of the control variable, equations (39), (40) and (41) will determine time paths of K_t, R_t and S_t. For convenience in calculation, this can be set up in a spreadsheet with rows for periods and columns for the variables.

To develop this into a method for finding a solution, we need a) to identify the corresponding difference equations for periods when $\nu > 0$ and specify when those equations apply, and b) to identify a criterion for optimality so that in choosing trial values of R_1 we can try values progressively closer to the optimum.

Since equations (40) and (41) are derived directly from the conditions of the problem, they apply at all times. What changes when $\nu > 0$ is that (39) is replaced by the following which corresponds to part of (13):

$$R_{t+1} = C^{1/\beta} A^{-1/\beta} K_{t+1}^{-\alpha/\beta} \qquad (42)$$

Using the condition (29), we can combine (39) and (42) as follows:

$$R_{t+1} = R_t + \frac{R_t}{1-\beta}\left(\delta(1-\alpha) - \frac{\alpha C}{K_{t+1}}\right) \text{ when } S(n) > \frac{C^{(1-\beta)/\beta}}{\alpha A^{1/\beta}(K(n))^{(\alpha-\beta)/\beta}} \equiv SC(n)$$

$$\text{and } = C^{1/\beta} A^{-1/\beta} K_{t+1}^{-\alpha/\beta} \text{ otherwise} \qquad (43)$$

To set this up in a spreadsheet it is useful to include a column for $SC(n)$.

It remains to identify a criterion for optimality. For this purpose it is useful to include in the spreadsheet a column for Y_t. On entering in the spreadsheet an initial trial value for R_1 and calculating through, it may be found that at some time the stock of resource has not been exhausted but $Y_t < C$. That cannot be optimal: indeed, it fails to meet the condition that output be sufficient to support consumption at rate C. In that case we need to increase the trial value of R_1.

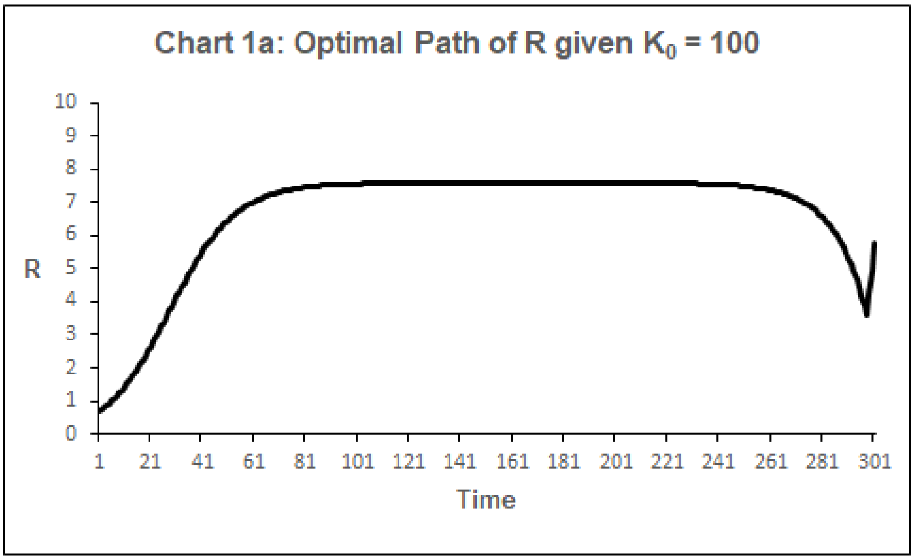
Alternatively, it may be found that $Y_t \geq C$ is always satisfied and that there is a time τ such that $Y_t = C$ whenever $t \geq \tau$, but that there is a large difference between $Y_{\tau-1}$ and Y_τ. More precisely, it may be found that $Y_{\tau-1} - Y_\tau$ is much larger than $Y_{\tau-2} - Y_{\tau-1}$. That cannot be optimal because, as shown by (31) within a continuous framework, optimality requires that Y approaches C as the time approaches τ. In that case we need to reduce the trial value of R_1.

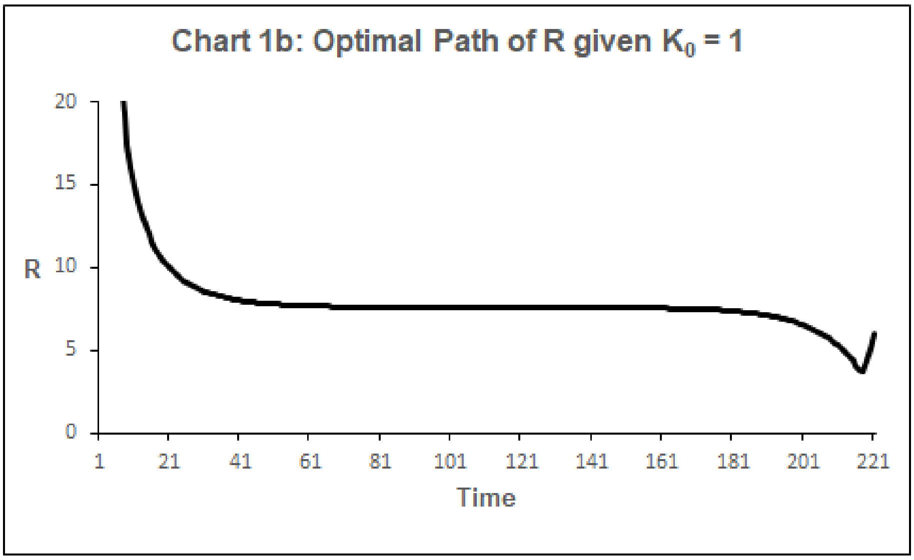
Depending on the parameters and initial values of the state variables, it may be found that the system is extremely sensitive to small adjustments in the trial value of R_1. In the case considered below it was necessary to go to 12 significant figures to obtain a satisfactory approximation to optimality.

Characteristics of an Optimal Solution

We have seen that, in the final stage of an optimal solution, the paths of K and R are determined respectively by (13) and (12), implying that K falls at the rate of depreciation while R rises so as to maintain output equal to required consumption while K falls. If the initial stocks $K(0)$ and $S(0)$ are such that condition (29) holds at $n = 0$, this "final" stage will be the whole solution. That will be the case when $S(0)$ is sufficiently small, although how small is sufficient will depend on $K(0)$, C and the parameters.

Prior to that final stage, in cases when condition (29) does not hold at $n = 0$, the paths of K and R are determined by (38) and (37), which allow those variables either to rise or to fall. When $S(0)$ is sufficiently large, it is found that the paths of the variables prior to the final stage can be divided into three phases. Charts 1a and 1b below show the optimal paths of R (obtained via the approximate method above) given the values $A = 1$; $\alpha = 0.3$; $\beta = 0.2$; $\delta = 0.1; C = 2$; $S_0 = 2000$, with two different values of K_0. Note that the vertical scale of Chart 1b, though much larger than that of Chart 1b, has been truncated with the effect that the path of R over the first few periods is not shown (eg $R_1 \approx 114$).





Both charts show a long central phase in which R remains approximately constant. During this phase, K and therefore Y also remain approximately constant. Using a term commonly employed in treatments of optimal growth and other contexts, we may describe this phase as a *turnpike*. Following the turnpike in both cases is a phase in which R and K fall, with Y therefore falling until it reaches the point at which $Y = C$ and the system enters the final stage in which R rises again (appearing as just a small uptick in the charts).

In both cases, the approximately constant values during the turnpike phase are the same: about 7.5 for R and 8.5 for K. But the phase prior to the turnpike differs according to the quantity of initial capital. If this is large, as in Chart 1a, R is initially small, limiting Y so that depreciation lowers K towards the turnpike level. If initial capital is small as in Chart 1b, on the other hand, R and Y are initially large, raising K towards the turnpike level. It is as if there is something about the turnpike which draws the system towards it.

To see what is special about the values of R and K in the turnpike, we first obtain, from (4), the condition on R for K to be constant:

$$\dot{K} = AK^{\alpha}R^{\beta} - C - \delta K = 0 \qquad (44)$$

$$AK^{\alpha}R^{\beta} = C + \delta K \qquad (45)$$

$$R = (C + \delta K)^{1/\beta}A^{-1/\beta}K^{-\alpha/\beta} \qquad (46)$$

Now we will find the value of K that minimises R, consistently with (46). Why? Because, given a finite quantity of the resource, the way to maximise the time over which required consumption can be maintained is to minimise its rate of use. Setting the derivative equal to zero:

$$\frac{dR}{dK} = A^{-1/\beta}\left[(C + \delta K)^{1/\beta}(-\alpha/\beta)K^{-(\alpha+\beta)/\beta} + (\delta/\beta)(C + \delta K)^{(1-\beta)/\beta}K^{-\alpha/\beta}\right] = 0 \qquad (47)$$

Dividing through by $A^{-1/\beta}(C + \delta K)^{1/\beta}K^{-\alpha/\beta}$:

$$(-\alpha/\beta)K^{-1} + (\delta/\beta)(C + \delta K)^{-1} = 0 \qquad (48)$$

$$-\alpha C - \alpha\delta K + \delta K = 0 \qquad (49)$$

$$K = \frac{\alpha C}{\delta(1 - \alpha)} \qquad (50)$$

To show that (50) corresponds to a minimum rather than a maximum of R we may note that (45) implies:

$$R^{\beta} = \frac{C}{A}K^{-\alpha} + \frac{\delta}{A}K^{1-\alpha} \qquad (51)$$

Given that $0 < \alpha < 1$, as K increases the first term on the right of (51) declines exponentially while the second increases exponentially. Hence the right hand side in total is convex and a point where its derivative with respect to K is zero must be a minimum.

Substituting the above parameter values into (50) and then substituting into (46) we obtain the following values to which K and R approximate in the turnpike phase:

$$K = \frac{0.3(2)}{0.1(1 - 0.3)} \approx 8.57 \qquad (52)$$

$$R = (2 + 0.1(8.57))^{1/0.2}1^{-1/0.2}8.57^{-0.3/0.2} = 2.857^{5}(8.57^{-1.5}) \approx 7.59 \qquad (53)$$

More generally, using (50) to substitute for (K) in (46), the approximate value of R in the turnpike phase is:

$$R = \Big(C + \frac{\alpha C}{1-\alpha}\Big)^{1/\beta} A^{-1/\beta} \Big(\frac{\alpha C}{\delta(1-\alpha)}\Big)^{-\alpha/\beta}$$

$$= \Big(\frac{C}{1-\alpha}\Big)^{1/\beta} A^{-1/\beta} \Big(\frac{\delta}{\alpha}\Big)^{\alpha/\beta} \Big(\frac{C}{1-\alpha}\Big)^{-\alpha/\beta}$$

$$= A^{-1/\beta} \Big(\frac{\delta}{\alpha}\Big)^{\alpha/\beta} \Big(\frac{C}{1-\alpha}\Big)^{(1-\alpha)/\beta} \qquad (54)$$

Considering progressively larger quantities of initial capital S_0 while holding initial capital and other parameters constant, the turnpike phase will occupy a larger and larger proportion of the total optimal path, and the other phases and final stage will become relatively insignificant. Thus the maximum time T will approximate closer and closer to S_0 divided by the right hand side of (54). Hence:

$$\lim_{S_0 \to \infty} \frac{T}{S_0} = A^{1/\beta} \Big(\frac{\alpha}{\delta}\Big)^{\alpha/\beta} \Big(\frac{1-\alpha}{C}\Big)^{(1-\alpha)/\beta} \qquad (55)$$

From this we can infer:

Proposition 2 *If S_0 is large then a reasonable approximation to the maximum duration T is given by:*

$$T = S_0 A^{1/\beta} \Big(\frac{\alpha}{\delta}\Big)^{\alpha/\beta} \Big(\frac{1-\alpha}{C}\Big)^{(1-\alpha)/\beta}$$

As an illustration, for our parameters as above the right hand side of (55) evaluates as below (we ignore the A term since $A = 1$):

$$\Big(\frac{0.3}{0.1}\Big)^{0.3/0.2} \Big(\frac{1-0.3}{2}\Big)^{(1-0.3)/0.2} = 3^{1.5} 0.35^{3.5} \approx 0.132 \qquad (56)$$

Our estimate from Proposition 1 for T given $S_0 = 1000$ and $K_0 = 100$ is therefore $1000(0.132) = 132$, whereas the actual T, obtained by our approximate method, is 170, a difference of 29%. For $S_0 = 2000$, again with $K_0 = 100$ (the case shown in Chart 1a), the estimate is $2000(0.132) - 264$, while the actual is 302, a difference of 14%. For $S_0 = 3000$, the estimate is $3000(0.132) = 396$ while the actual is 434, a difference of only 10%, illustrating the convergence of approximation to actual as S_0 becomes larger.

$* *$

Reference

Chiang A C (1999) *Elements of Dynamic Optimization* Waveland Press